Day 4: Motion Along a Curve — Vectors

I give my students the following list of terms and formulas to know.

Parametric Equations, Vectors, and Calculus — Terms and Formulas to Know:

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at the point (x, y) is given by $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ where $\dfrac{dx}{dt} \neq 0$, and the second derivative is given by $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$.

The derivative $\dfrac{dy}{dx}$ also may be interpreted as the slope of the tangent line to the curve C, or as the slope of the path of a particle traveling along the curve C, or as the rate of change of y with respect to x.

The second derivative $\dfrac{d^2y}{dx^2}$ is the rate of change of the slope of the curve C with respect to x.

$x'(t) = \dfrac{dx}{dt}$ is the rate at which the x-coordinate is changing with respect to t or the velocity of the particle in the horizontal direction.

$y'(t) = \dfrac{dy}{dt}$ is the rate at which the y-coordinate is changing with respect to t or the velocity of the particle in the vertical direction.

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$\langle x(t), y(t) \rangle$ is the **position vector** at any time *t*.
$\langle x'(t), y'(t) \rangle$ is the **velocity vector** at any time *t*.
$\langle x''(t), y''(t) \rangle$ is the **acceleration vector** at any time *t*.

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the **speed of the particle** or the **magnitude (length) of the velocity vector**.

$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is the **length of the arc (or arc length) of the curve from** $t = a$ to $t = b$ or the **distance traveled by the particle from** $t = a$ to $t = b$.

Most textbooks do not contain the types of problems on vectors that are found on the AP Exam, so I supplement with the examples and worksheets below.

Example 1 (no calculator):

A particle moves in the *xy*-plane so that at any time *t*, the position of the particle is given by $x(t) = t^3 + 4t^2,\ y(t) = t^4 - t^3$.

(a) Find the velocity vector when $t = 1$.

Solution:

$$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{d}{dt}\left(t^3 + 4t^2\right), \frac{d}{dt}\left(t^4 - t^3\right) \right\rangle = \left\langle 3t^2 + 8t, 4t^3 - 3t^2 \right\rangle$$

$$v(1) = \langle 11, 1 \rangle$$

(b) Find the acceleration vector when $t = 2$.

Solution:

$$a(t) = \left\langle \frac{d}{dt}\left(\frac{dx}{dt}\right), \frac{d}{dt}\left(\frac{dy}{dt}\right) \right\rangle = \left\langle \frac{d}{dt}\left(3t^2 + 8t\right), \frac{d}{dt}\left(4t^3 - 3t^2\right) \right\rangle = \left\langle 6t + 8, 12t^2 - 6t \right\rangle$$

$$a(2) = \langle 20, 36 \rangle$$

Example 2 (no calculator):

A particle moves in the *xy*-plane so that at any time *t*, $t \geq 0$, the position of the particle is given by $x(t) = t^2 + 3t,\ y(t) = t^3 - 3t^2$. Find the magnitude of the velocity vector when $t = 1$.

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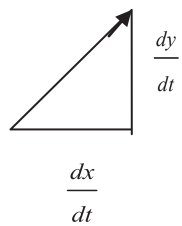
Solution:

The magnitude or length of the velocity vector can be found by using the Pythagorean Theorem, since the horizontal and vertical components make a right triangle, with the vector itself as the hypotenuse. Therefore its length is given by:

$$\text{Magnitude of velocity vector} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$



For our problem, $\frac{dx}{dt} = \frac{d}{dt}\left[t^2 + 3t\right] = 2t + 3$ and $\frac{dy}{dt} = \frac{d}{dt}\left[t^3 - 3t^2\right] = 3t^2 - 6t$.

$$\text{Magnitude of velocity vector} = \sqrt{(2t+3)^2 + (3t^2 - 6t)^2}\,\Bigg|_{t=1} = \sqrt{25+9} = \sqrt{34}.$$

Notice that the formula for the magnitude of the velocity vector is the same as the formula for the speed of the vector, which makes sense since speed is the magnitude of velocity.

Example 3 (no calculator):

A particle moves in the *xy*-plane so that
$x = \sqrt{3} - 4\cos t$ and $y = 1 - 2\sin t$, where $0 \le t \le 2\pi$.

The path of the particle intersects the *x*-axis twice. Write an expression that represents the distance traveled by the particle between the two *x*-intercepts. Do not evaluate.

Solution:

The path of the particle intersects the *x*-axis at the points where the *y*-component is equal to zero. Note that $1 - 2\sin t = 0$ when $\sin t = \frac{1}{2}$. For $0 \le t \le 2\pi$, this will occur when

$t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. Since $\frac{dx}{dt} = \frac{d}{dt}\left[\sqrt{3} - 4\cos t\right] = 4\sin t$ and $\frac{dy}{dt} = \frac{d}{dt}\left[1 - 2\sin t\right] = -2\cos t$,

the distance traveled by the particle is Distance = $\int_{\pi/6}^{5\pi/6} \sqrt{(4\sin t)^2 + (-2\cos t)^2}\, dt$.

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Day 4 Homework

Use your calculator on problems 10 and 13c only.

1. If $x = t^2 - 1$ and $y = e^{t^3}$, find $\dfrac{dy}{dx}$.
2. If a particle moves in the xy-plane so that at any time $t > 0$, its position vector is $\langle \ln(t^2 + 5t), 3t^2 \rangle$, find its velocity vector at time $t = 2$.
3. A particle moves in the xy-plane so that at any time t, its coordinates are given by $x = t^5 - 1$ and $y = 3t^4 - 2t^3$. Find its acceleration vector at $t = 1$.
4. If a particle moves in the xy-plane so that at time t its position vector is $\left\langle \sin\left(3t - \dfrac{\pi}{2}\right), 3t^2 \right\rangle$, find the velocity vector at time $t = \dfrac{\pi}{2}$.
5. A particle moves on the curve $y = \ln x$ so that its x-component has derivative $x'(t) = t + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point (1, 0). Find the position of the particle at time $t = 1$.
6. A particle moves in the xy-plane in such a way that its velocity vector is $\langle 1 + t, t^3 \rangle$. If the position vector at $t = 0$ is $\langle 5, 0 \rangle$, find the position of the particle at $t = 2$.
7. A particle moves along the curve $xy = 10$. If $x = 2$ and $\dfrac{dy}{dt} = 3$, what is the value of $\dfrac{dx}{dt}$?
8. The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - \dfrac{3}{2}t^2 - 18t + 5$ and $y = t^3 - 6t^2 + 9t + 4$. For what value(s) of t is the particle at rest?
9. A curve C is defined by the parametric equations $x = t^3$ and $y = t^2 - 5t + 2$. Write the equation of the line tangent to the graph of C at the point (8, –4).
10. A particle moves in the xy-plane so that the position of the particle is given by $x(t) = 5t + 3\sin t$ and $y(t) = (8 - t)(1 - \cos t)$. Find the velocity vector at the time when the particle's horizontal position is $x = 25$.
11. The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 3$ and $y(t) = \dfrac{2}{3}t^3$.

    (a) Find the magnitude of the velocity vector at time $t = 5$.

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(**b**) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

(**c**) Find $\dfrac{dy}{dx}$ as a function of x.

12. Point $P(x, y)$ moves in the xy-plane in such a way that $\dfrac{dx}{dt} = \dfrac{1}{t+1}$ and $\dfrac{dy}{dt} = 2t$ for $t \ge 0$.

(**a**) Find the coordinates of P in terms of t given that, when $t = 1$, $x = \ln 2$ and $y = 0$.

(**b**) Write an equation expressing y in terms of x.

(**c**) Find the average rate of change of y with respect to x as t varies from 0 to 4.

(**d**) Find the instantaneous rate of change of y with respect to x when $t = 1$.

13. Consider the curve C given by the parametric equations $x = 2 - 3\cos t$ and $y = 3 + 2\sin t$, for $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$.

(**a**) Find $\dfrac{dy}{dx}$ as a function of t.

(**b**) Find the equation of the tangent line at the point where $t = \dfrac{\pi}{4}$.

(**c**) The curve C intersects the y-axis twice. Approximate the length of the curve between the two y-intercepts.

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